Math 131 Final Exam - Solutions

Fall 2021 - December 15th, 2021

Name: ____________________

Instructor Name: ____________________

Did you have another exam 5:30-7:30 today (December 15). Circle: Yes No

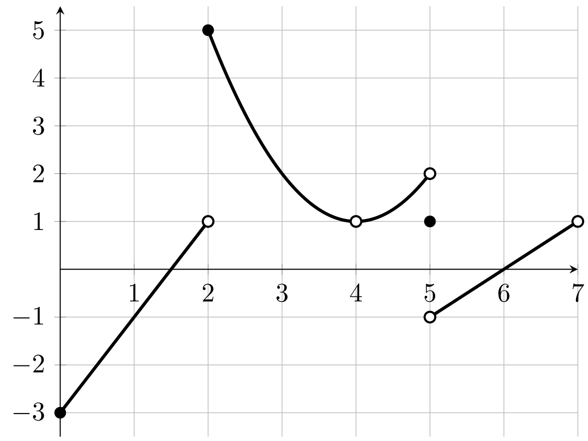
Page:	1	2	3	4	5	6	7	8	9	Total
Points:	20	8	18	22	9	25	22	18	10	152
Score:										

Answer the questions in the spaces provided on the question sheets. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but also how you obtained it. Include units in your answer when possible. You may receive 0 points for a problem where you show no work.

Instructions:

1. Do not open this exam until told to do so.
2. No books or notes may be used on the exam. There is an equation sheet on the last page of this exam.
3. Credit or partial credit will be given only when the appropriate explanation and/or algebra is shown.
4. Make sure your answer is clearly marked.
5. Read and follow directions carefully.
6. This exam has 14 questions, for a total of 152 points. There are 9 pages. Make sure you have them all.
7. You will have 120 minutes to complete the exam.
8. All cell phones and electronic devices (other than calculators) must be turned off during the exam.
9. Do not separate any of the pages of this exam except the last one containing the equation sheet. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
10. Calculators without internet access are allowed.
11. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.

1. (12 points) Using the graph below, find each of the following. If the answer does not exist, write "DNE".



(a) $f(1)$

(b) $f(4)$

(c) $f(5)$

(d) $\lim_{x\to 2} f(x)$

(e) $\lim_{x\to 4} f(x)$

(f) $\lim_{x\to 5^-} f(x)$

Solution: Each part is worth 2 points.

(a) -1

(b) DNE

(c) 1

(d) DNE

(e) 1

(f) 2

2. Find each of the following limits exactly. If the answer does not exist, write "DNE". Use L'Hopital's Rule if it applies

(a) (4 points) $\lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

Solution:

$$\lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x\to 2} \frac{x(x-2)(x-5)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x(x-5)}{(x+3)} = \frac{-6}{5}$$

- Up to 2 points for their algebra
- 2 points for correct answer

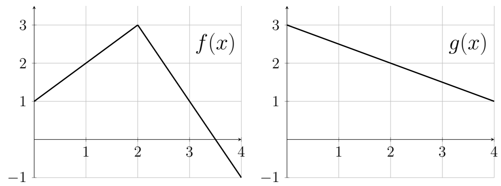
(b) (4 points) $\lim_{x\to\infty} \frac{x}{e^x}$

Solution:

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

- 1 point for attempting L'Hopital's rule.
- 1 point for correct derivatives
- 2 points for correct answer

3. Use the graphs of $f(x)$ and $g(x)$ below to compute the derivatives. If the answer does not exist, write "DNE".



(a) (4 points) If $h(x) = f(x)g(x)$, compute $h'(1)$.

Solution: $h'(x) = f'(x)g(x) + f(x)g'(x)$ so $h'(1) = 1 \cdot 2.5 + 2 \cdot -0.5 = 1.5$

- 1 point for writing out $h'(x)$ correctly.
- 0.5 point each for substituting the correct values for $f(1)$ and $f'(1)$
- 0.5 point each for substituting the correct values for $g(1)$ and $g'(1)$
- 1 point for correctly calculating the answer.

(b) (4 points) If $j(x) = f(g(x))$, compute $j'(1)$.

Solution: $j'(x) = f'(g(x)) \cdot g'(x)$ so $j'(1) = f'(2.5) \cdot -0.5 = -2 \cdot -0.5 = 1$

- 1 point for writing out $j'(x)$ correctly.
- 0.5 point each for substituting the correct values for $f(1)$ and $f'(1)$
- 0.5 point each for substituting the correct values for $g(1)$ and $g'(1)$
- 1 point for correctly calculating the answer.

4. Find $\frac{dy}{dx}$ for the following; you do not have to simplify.

(a) (4 points) $y = x^2 \cdot 2^x - 3$

Solution: $y' = (2x)(2^x) + \ln(2)x^2 \cdot 2^x$. No more than 4 points off

- 1 point off if the derivative of the constant is not zero.
- 2 points off for not using the product rule or an incorrect rendition of the product rule.
- 1 point off for incorrectly taking the derivative of x^2.
- 1 point off for incorrectly taking the derivative of 2^x.

(b) (4 points) $y = \frac{e^x}{\cos(x)}$

Solution: $y' = \frac{e^x \cos(x) + e^x \sin(x)}{\cos^2(x)} = \frac{e^x(\cos(x)+\sin(x))}{\cos^2(x)}$. No more than 4 points off

- 2 points off for not using the quotient rule or for not rewriting it as $e^x(\cos(x))^{-1}$ or for an incorrect rendition of the quotient rule
- 1 point off for incorrectly taking the derivative of e^x
- 1 point off for incorrectly taking the derivative of $\cos(x)$
- 1 point off for not squaring the denominator

5. For the function $f(t) = \frac{t}{1+t^2}$

(a) (10 points) Find critical points and perform the first derivative test to find local minima/maxima. *You only need to provide the t-value where the max/min occur.*

Solution:

$$f'(t) = \frac{1 + t^2 - 2t^2}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2} = \frac{(1-t)(1+t)}{(1+t^2)^2}$$

The critical points are at $t = \pm 1$. By the sign analysis of $f'(t)$ we know that $f(t)$ is increasing on the interval $(-1, 1)$ and decreasing on the intervals $(-\infty, -1) \cup (1, \infty)$. Therefore by the first derivative test we have that $f(t)$ has a local minimum at $t = -1$ and a local maximum at $t = 1$.

- Up to 2 points for the derivative
- 2 points for setting it equal to zero
- Up to 2 points for finding the zeros
- Up to 2 points doing a sign analysis
- 2 point for the correct answer(s)

(b) (6 points) The second derivative is $f''(t) = \dfrac{2t^3 - 6t}{(t^2+1)^3}$. Find the inflection points for $f(t)$. Make sure to verify that they are inflection points. *You only need to provide the t-value where the inflection points occur.*

Solution: The second derivative $f''(t) = \dfrac{2t(t^2-3)}{(t^2+1)^3}$ is zero at $t = 0, \pm\sqrt{3}$ and is never undefined. By inspection we see that the second derivative actually changes signs at all three values so they are all inflection points.

- Up to 2 points for each answer with proper justification.
- Note, the instructions allow for graphing of the function, so if they sketch the graph or make a note of using the calculator this would be sufficient.

(c) (4 points) Find the global maximum and minimum for $f(t)$ on the closed interval $[0, 5]$.

Solution: Evaluating at the critical points and end points we see that $f(0) = 0, f(5) = \frac{5}{26}$, and $f(1) = \frac{1}{2}$. Therefore the global maximum is at $t = 1$ and the global minimum is at $t = 0$.

- 1 point each for a correct answer
- 1 point each for justification of correct answer. (It is sufficient to list the 3 values or reference looking at the graph.)

6. (12 points) Provide a careful sketch of a graph of a single function $f(x)$ that satisfies the following six conditions. No formula is needed just carefully sketch and label your graph. Full credit will not be given for a graph that is not carefully labeled or that does not clearly satisfy the six conditions indicated.

i. $f(0) = 2$;

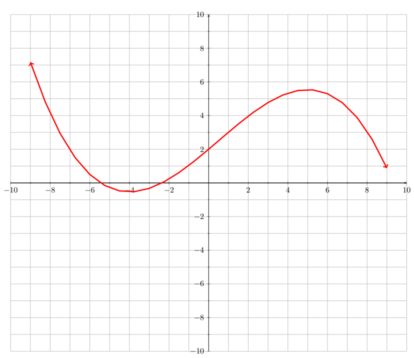
ii. $f(x)$ is continuous;

iii. $f'(x) < 0$ for $x < -4$ and for $x > 5$;

iv. $f'(x) > 0$ for $-4 < x < 5$;

v. $f''(x) > 0$ for $x < 0$;

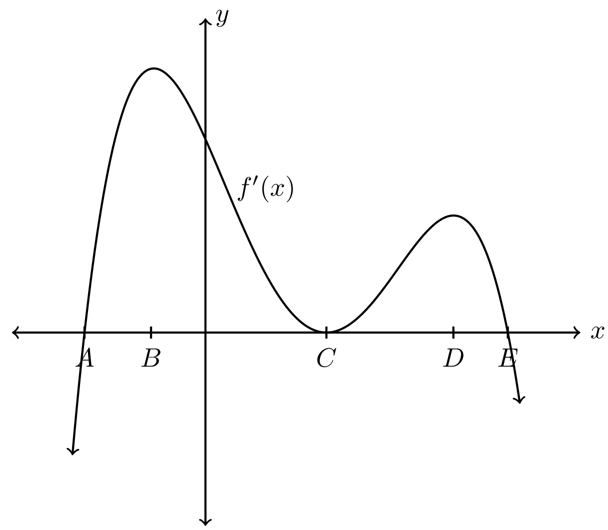
vi. $f''(x) < 0$ for $0 < x$.



Solution: There are many possible solutions

- Up to 2 points for each condition. Partial credit of 1pt is allowed.

7. Consider the graph of the **DERIVATIVE** $f'(x)$ given below.



(a) (3 points) Identify all the critical points of the **function** $f(x)$.

> **Solution:** The critical points of $f(x)$ are when $f'(x) = 0$ which happens when $x = A, C, E$.
>
> - Award 1 point each for A, C and E.
>
> No work or explanation is required.

(b) (3 points) Classify the critical points you found in part (a) as local minimum, local maximum, or neither.

> **Solution:** The x-value $x = A$ corresponds to a local minimum, $x = C$ is neither, and $x = E$ corresponds to a local maximum.
>
> - Award 1 points each for each correct classification.
>
> No work or explanation is required. If they added extra points in part (a) then they were already penalized.

(c) (3 points) Find all the inflection points of the **function** $f(x)$.

> **Solution:** The inflection points are at $x = B, C, D$
>
> - Award 1 point each for B, C and D.
>
> No work or explanation is required and they do not need to provide additional verification that they are inflection points.

8. A manufacturer of baseball bats makes x bats at a cost of $C(x) = 4x + 10$ dollars. The revenue from the sale of x bats is given by $R(x) = 50x - 0.5x^2$ dollars.

   (a) (6 points) How many bats should be manufactured and sold to maximize profit?

   Solution: We maximize profit by setting marginal revenue equal to marginal cost. So $R'(x) = C'(x) \Rightarrow 50 - x = 4$ happens when $x = 46$.

   - 3 points for setting it up correctly. Either $\pi'(x) = R'(x) - C'(x)$ or $R'(x) = C'(x)$
   - 3 points for finding the correct value

   (b) (4 points) Use the second derivative to show that the number of bats you found gives a maximum profit.

   Solution: The second derivative of profit is $\pi''(x) = -1$. Therefore the profit function is always concave down and at the critical point $x = 46$ we have a local maximum.

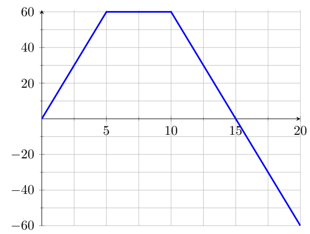
   - Award 1 point for knowing $\pi(x) = R(x) - C(x)$
   - Award 2 points for finding the second derivative of profit.
   - Award 1 point for stating that profit is concave down.

9. (15 points) A grocery store wants to replace paper bags with a sturdy open top container, with a square base, that holds 4 ft^3 of groceries. What should the dimensions of the container be in order to use the least amount of material? Start by drawing a diagram

   Solution: The surface area of the box is $A = x^2 + 4xh$. The volume of the box is $V = x^2h$ so this implies that $h = \frac{4}{x^2}$. Plugging this into the surface area equation yields $A = x^2 + \frac{16}{x}$, which has a derivative of $A' = 2x - \frac{16}{x^2}$. This gives us critical points of $x = 0$ and $x = 2$. The value of $x = 0$ doesn't make sense for this problem. The second derivative of the surface area is $A'' = 2 + \frac{32}{x^3}$ which is postive at $x = 2$ so by the second derivative test $x = 2$ gives a minimum for surface area. When $x = 2$ we know $h = 1$. The final dimensions of our box will be 2 ft. $\times$ 2 ft. $\times$ 1 ft.

   - 2 pts. for drawing and correctly labeling a diagram.
   - 2 pts. for the correct equation for volume
   - 2 pts. for the correct equation for surface area
   - 2 pts. for substituting the correct substitution for height
   - 2 pts. for the correct derivative
   - 2 pts. for the correct value of x
   - 2 pts. for the correct value of height
   - 1 pt. for either a sign analysis of the first or second derivative

10. This graph shows the velocity, in km/hr of a drone flying over a 24 hour period. Distance is measured in km away from its starting base. At time $t = 0$, the distance traveled is 0. Time is measured in hours and the velocity is measured in km per hour.



(a) (8 points) Complete the table for the **distance** the drone is from its starting base after t hours.

Time (hrs)	0	5	10	15	20
Distance (km)	0	150	450	600	450

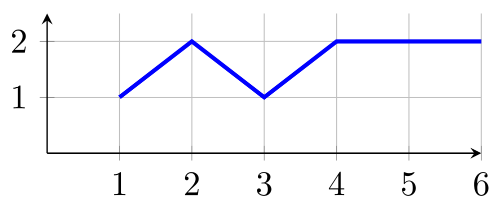
Solution: 2 points each cell (all or nothing per cell). However, if you notice that a single mistake for a particular region compounds you can only mark one cell incorrect.

(b) (4 points) On what interval(s) is the **distance** from the base decreasing?

Solution: The distance is decreasing when the derivative is negative. This happens when $15 < t \leq 20$.

- 4 points for the correct interval (all or nothing)

11. The graph of $f(x)$ is give below.



(a) (5 points) Use the figure to compute $\int_1^6 f(x)\,dx$.

Solution: Computing areas we see that $\int_1^6 f(x)\,dx = 8.5 = \frac{17}{2}$.

- 5 points – all or nothing

(b) (5 points) What is the average value of f on $[1, 6]$?

Solution: The average value is $\frac{1}{6-1}\int_1^6 f(x)\,dx = \frac{8.5}{5} = \frac{17}{10}$

- 3 points for setting it up correctly.
- 2 points for the correct answer.

12. Find these indefinite integrals (don't forget the $+C$ where appropriate)

(a) (4 points) $\int \left(x^4 + \frac{7}{x} + \frac{8}{x^3}\right)\, dx$

Solution: $\int \left(x^4 + \frac{7}{x} + \frac{8}{x^3}\right)\, dx = \frac{x^5}{5} + 7\ln|x| - \frac{4}{x^2} + C$

- Deduct 1 point for each missing +C
- Deduct 1 point for each incorrect term.

(b) (4 points) $\int (7\cos(x) + 3e^x)\, dx$

Solution: $\int (7\cos(x) + 3e^x)\, dx = 7\sin(x) + 3e^x + C$

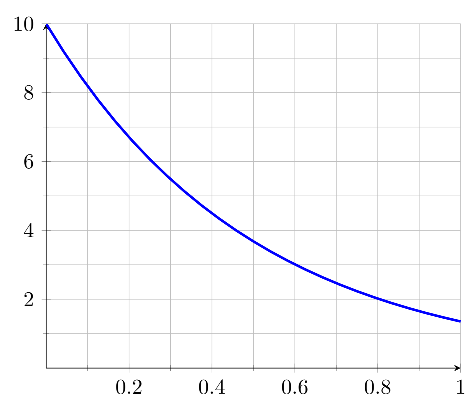
- Deduct 1 point for each missing +C
- Deduct 1 point for each incorrect term.

(c) (4 points) $\int 200(1.03)^x\, dx$

Solution: $\int 200(1.03)^x\, dx = \frac{200}{\ln 1.03}(1.03)^x + C$

- Deduct 1 point for each missing +C
- Deduct 1 point for each incorrect term.
- 1 point for writing the 200, 1 point for writing 1.03^x 1 point for dividing by $\ln 1.03$

13. (a) (6 points) Use a left Riemann sum and $n = 5$ rectangles to approximate $\int_0^1 10e^{-2x}\, dx$. Draw the rectangles and show your work.



Solution: The approximate answer is $0.2(10 + 6.8 + 4.5 + 3 + 2) = 5.26$
No more than 6 points off

- 1 point off for each incorrectly drawn the rectangles.
- 1 point off for having the incorrect Δx

- 1 point off for each incorrect, missing or unreasonable value(s) of $f(x)$
- 1 point off for not having the correct sum

(b) (6 points) Evaluate $\int_0^1 10e^{-2x}\,dx$ exactly.

Solution: The anti derivative of $f(x) = 10e^{-2x}$ is $F(x) = -5e^{-2x}$ so by the Fundamental Theorem of Calculus we have

$$\int_0^1 10e^{-2x}\,dx = F(1) - F(0) = -5e^{-2} + 5 \approx 4.3233$$

- 3 points for the correct anti-derivative
- 1 point each for substituting in 1 and 0 correctly
- 1 point for the correct exact answer.

14. (4 points) The concentration of a medication in the plasma changes at a rate of $h(t)$ mg/ml per hour, t hours after the delivery of the drug. There is 250 mg/ml of the medication present at time $t = 0$ and $\int_0^3 h(t)\,dt = 480$. Determine the plasma concentration of the medication present three hours after the drug is administered. *Make sure to include units in your answer.*

Solution: If $H(t)$ is an antiderivative of $h(t)$ then by the FTC we have that $H(3) = H(0) + \int_0^3 h(t)\,dt$. Now $H(0) = 250$ and so we see that $H(3) = 250 + 480 = 730$ mg / ml.

- 2 points for using FTC
- 1 point for the correct value.
- 1 point for the correct units.